Liquid Crystals
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713926090

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Online publication date: 06 August 2010

To cite this Article Pauwels, H. and Lagerwall, S. T.(2001) 'Analytical approach to grey levels in ferroelectric liquid crystal displays', Liquid Crystals, 28: 4, 573 — 580
To link to this Article: DOI: 10.1080/02678290010017971
URL: http://dx.doi.org/10.1080/02678290010017971

LIQUID CRYSTALS, 2001, VOL. 28, NO. 4, 573–580

Analytical approach to grey levels in ferroelectric liquid crystal displays

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(*Received 9 July 2000; in final form 5 September 2000; accepted 30 September 2000*)

Grey levels in ferroelectric liquid crystal displays can be obtained through 'continuous director rotation' in monostable modes. Displays can only be actively addressed. A very simple description of this continuous director rotation can be given in the so-called uniform φ theory, where it is assumed that the azimuthal angle φ of the director does not vary throughout the thickness of the liquid crystal. It requires however that the equilibrium state at zero voltage corresponds to the director making a maximum angle, i.e. the cone angle, with the glass substrates, the polarization being parallel to the glass substrates. It is shown that the electro-optic response can be given an analytic treatment which reduces to the uniform φ theory with the appropriate zero voltage equilibrium state. It accounts entirely for the observed V-shaped switching in high-**P** materials. In particular an exact expression for the saturation voltage and its dependence on the **P** value and on the alignment layer thickness is given.

1. Introduction

Ferroelectric liquid crystal displays usually switch between two stable states corresponding to dark and bright. Obtaining analogue grey levels is therefore a difficult task. Two solutions have been proposed: in-pixel domain switching and continuous director rotation. In the first solution parts of a pixel switch from the dark to the bright state or vice versa, because the switching parameters, either static or dynamic, vary over the area of the pixel. In antiferroelectric liquid crystal displays, this principle works well. The switching voltages were of the order of 20 V; it allows passive matrix addressing. In ferroelectric liquid crystal displays using passive matrix addressing, it has not worked, mainly because the static threshold is too low and hence the switching is too slow [1]. Recently it was proposed to use the dependence of the dynamic threshold on the chevron angle to cause a controlled variation of this threshold over the area of a pixel [2]. This works for grey levels but destroys the contrast because no perfectly dark state is possible.

The second solution, continuous director rotation, is based on the existence of stationary states between the dark and bright states, and thus produces grey levels. These stationary states are thresholdless, have no hysteresis, are voltage-controlled and can only be addressed actively. But since active matrices have now become a mature technology, this is no longer considered to be a disadvantage. Several models and technologies have been proposed: the V-shaped antiferroelectric model [3], the V-shaped ferroelectric model [4], the Sony mode and the continuous director rotation (CDR) mode [5]. Independently of any proposed models, some excellent grey level display prototypes have been demonstrated [6, 7].

In the present paper, we want to develop a very simple theory which is based on describing the stationary state as a so-called uniform state. This means that the orientation of the director is described by an azimuthal angle φ which is uniform throughout the thickness of the liquid crystal. The validity of this simple theory will be proved by the reduction of the exact analytic theory to this simple theory. The theory is developed for the bookshelf structure because the (relatively minor) complication due to the unavoidable chevron structure can be taken into account seperately. In fact, the presence of chevrons has a beneficial influence on the optical properties (making the cell more symmetrical at oblique incidence), whereas it has little influence on the qualitative

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Liquid Crystals ISSN 0267-8292 print/ISSN 1366-5855 online © 2001 Taylor & Francis Ltd
http://www.tandf.co.uk/journals
DOI: 10.1080/02678290010017971

aspects of the electrical analysis given below. The reference for the angle φ is chosen such that $\varphi = 0$ or π means that the permanent polarization is perpendicular to the glass plates and the director parallel to the glass plates, as shown in figure 1.

In [8] this description was applied to antiferroelectric liquid crystals. Antiferroelectric liquid crystals are described by two angles φ_1 and φ_2 [9], but the problem could be reduced to a single angle $\varphi_1 = -\varphi_2$. In [10] the formalism was applied to low-**P** ferroelectric liquid crystals. In this paper the analytic theory is presented as corresponding to the physical model given in [4] and further discussed in [11] and [12]. The starting point is the observation that the electrostatic stiffening of the splay and bend elastic constants, together with polar boundary conditions, will lead to a uniform director alignment at high **P** values.

2. Uniform theory

The configuration of a ferroelectric liquid crystal (FLC) in a bookshelf structure is shown in figure 1. The uniform theory consists of assuming that the azimuthal angle φ is independent of the coordinate perpendicular to the glass plates. A state of the FLC is then described by the single parameter φ. This φ is determined by the minimization of an energy density expression

$$W = -\mathbf{EP}\cos\varphi - \frac{\gamma}{d}\sin^2\varphi. \tag{1}$$

The first term expresses the tendency of **P** to align with **E**. The second term, with $\gamma > 0$, describes an interaction between the liquid crystal and the alignment layers (reduced to an energy density). The sign of this term expresses that at $V = 0$, the $\varphi = \pm\pi/2$ state is a preferred state, which means that the director is then not parallel with the glass plates but makes a maximum angle (i.e. the cone angle) with the glass plates. Usually the interaction with the alignment layer causes the director to be parallel to the glass plates, requiring the positive sign for this second term. In each continuous director rotation model we shall need some explanation to justify the negative sign. Why the sign has to be negative will be shown in what follows.

A stationary state minimizes W, and therefore obeys

$$\frac{\partial W}{\partial\varphi} = \mathbf{EP}\sin\varphi - \frac{\gamma}{d}\sin 2\varphi = 0. \tag{2}$$

This equation is solved graphically in figure 2 which plots the $\sin\varphi$ and $\sin 2\varphi$ curves in equation (2). The intersections are the stationary solutions. There are three solutions: $\varphi = 0$, $\varphi = \pi$ and an intermediate solution described by

$$\cos\varphi = \frac{V\mathbf{P}}{2\gamma} \tag{3}$$

where V is the voltage over the FLC. Notice that if φ is uniform, also **E** in the FLC is a constant and $V = \mathbf{E}d$. These solutions are however not necessarily stable, i.e. correspond to a minimum of energy. Stability requires that $\partial W/\partial\varphi$ goes from negative through zero to positive with increasing φ as shown in the inset. The sign of $\partial W/\partial\varphi$ is shown by the arrow on figure 2: positive means the arrow pointing upwards. The intermediate solution is only stable—i.e. corresponds to a minimum of W—if the sign of the second term in equation (1) is negative, so that $\partial W/\partial\varphi$ goes from negative through zero to positive with increasing φ. This intermediate stable solution varies continuously from $\varphi = \pi/2$ for $V = 0$ to $\varphi = 0$ for $V = V_m = 2\gamma/\mathbf{P}$, or to $\varphi = \pi$ for $V = -2\gamma/\mathbf{P}$. The $\sin\varphi$-curve for $V = \pm V_m$ is shown by the dotted line. For values of V outside this range, the intermediate solution ceases to exist. For $V > 2\gamma/\mathbf{P}$ the solution $\varphi = 0$ is stable, and the solution $\varphi = \pi$ is stable for $V < -2\gamma/\mathbf{P}$.

If the sign of the second term of equation (1) is positive, the intermediate solution is unstable. If for

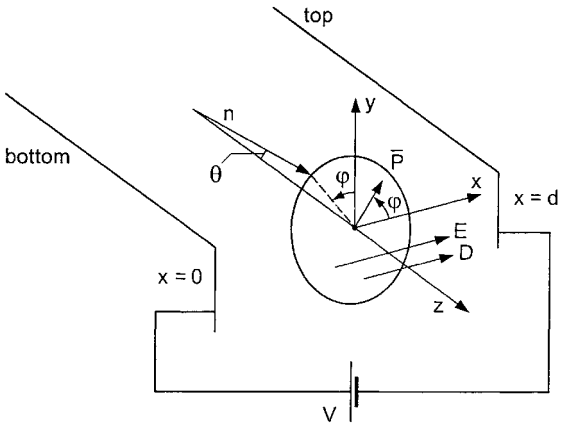

Figure 1. Configuration of the ferroelectric liquid crystal cell.

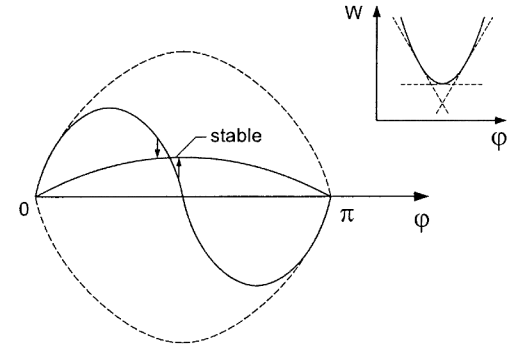

Figure 2. Graphical determination of stable states. The arrow indicates the sign of $\partial W/\partial\varphi$. The arrow pointing upwards means $\partial W/\partial\varphi > 0$. The dotted lines describe the situation for $V = \pm V_m$. The inset picture illustrates the condition for stability.

increasing V we start out from the solution $\varphi = \pi$, we jump over to $\varphi = 0$ at $V = +V_m$, and for decreasing V we jump back to $\varphi = \pi$ at $V = -V_m$. We obtain a threshold and hysteresis, i.e. the normal behaviour of ferroelectric liquid crystal displays. The 'continuous director rotation mode' requires the negative sign.

The transmission in a uniform situation is for a proper orientation of the crossed polarizer and analyser, given by the standard formula [10]

$$T = \frac{1}{2}\sin^2 2(\theta - \psi_{tw})\sin^2\left[\frac{\pi}{2}(1 - \sin^2\theta\sin^2\varphi)\right] \tag{4}$$

with $\tan\psi_{tw} = \cos\varphi\tan\theta$. This twist angle ψ_{tw} measures the orientation of the projection of the director on the substrates. Figure 3 shows the transmittance given by formula (4) versus the voltage.

The uniform φ theory is thus simple and straightforward, but unfortunately it is not correct. The stationary states in FLCs cannot in general be considered uniform and the interaction with the alignment layer cannot always be described by equation (1). Especially, there must be a physical justification of the negative second term in equation (1). For the V-shaped AFLC, the negative term could be justified by the presence of transverse electrical dipoles in high-**P** material [8]. Although AFLCs are described by two azimuthal angles, for the so-called symmetrical up or down states $\varphi_1 = -\varphi_2$ amounts to a single angular variable φ. Assuming the uniform theory gives, in the end, exactly the previous equations. In [10], for the CDR mode of low-**P** ferroelectric material, the effective negative term in equation (1) is caused by the dielectric torque for sufficiently large **E**. In [4] the V-shaped characteristic is explained as being related to the splayed state in high-**P** ferroelectric materials. We will in the following treat this case analytically and show that it can be reduced, at least qualitatively, to the uniform theory.

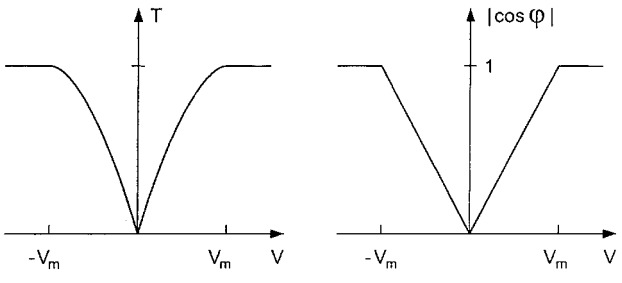

Figure 3. Typical V-shaped T–V characteristic and related $\cos\varphi$ dependence.

3. Stationary states in the splayed ferroelectric liquid crystal

A quantitative analysis must determine the electrostatic field distribution in the liquid crystal. It is well known that there are two equivalent ways of looking at this problem. Either one considers the field distribution as caused by the extra polarization charge in free space, i.e. one considers the equation $\nabla\ \varepsilon\varepsilon_0\mathbf{E} = -\nabla\ \mathbf{P}$, or one introduces the electric induction $\mathbf{D} = \varepsilon\varepsilon_0\mathbf{E} + \mathbf{P}$ with $\nabla\ \mathbf{D} = 0$. The first approach is used in several recent publications [4, 11, 12] where the main emphasis is on qualitative physical arguments to explain why the strong polar boundary conditions in the splayed ferroelectric liquid crystal will untwist the bulk.

In the present article we will use the second approach because it leads in a straightforward way to quantitative results. The equation $\nabla\ \mathbf{D} = 0$ indeed reduces to the simple result that **D** is a constant throughout the liquid crystal, including the alignment layers. We shall first analyse the liquid crystal and finally add the voltage drop over the alignment layers. Because of the high values of **P**, the voltage drop over the alignment layers will normally be more important than the voltage over the liquid crystal. The presence of a chevron structure (with a very small chevron angle) is neglected. In as far as the continous theory is valid, the analysis is exact.

The equation of director motion for ferroelectric switching (i.e. when **E** is not taken as a constant) is treated by several authors. Pauwels and co-workers [13, 14] follow the approach explained by Nakagawa and co-workers [15, 16], in which the free energy is expressed as a function of the director orientation and of the electric potential. Variation of the director orientation leads to its equation of motion and variation of the potential to Maxwell's equations. Towler and co-workers [17] reformulate the free energy using Maxwell's equations to depend only on the director orientation. This free energy is then minimized. Pauwels has shown in [14] that both approaches lead to the correct result. We shall follow the Pauwels approach, because it is usually simpler. The expression for the Gibbs free energy is

$$G = F_d + F_s + \int_V \left[-\mathbf{E}\ \mathbf{P} - \frac{1}{2}\varepsilon_0\varepsilon\mathbf{E}^2\right]dv$$

where F_d is the distortion energy and F_s the surface interaction energy, and where we have neglected the dielectric anisotropy. We indeed think that dielectric anisotropy is not essential in this high-**P** problem. Towler and co-workers [17] apply the non-uniform theory for investigating switching at high voltages and there it is known that the dielectric anisotropy plays an essential role.

In the so-called one constant approximation for the distortion energy ($K = K_1 = K_2 = K_3$), stationary states are given [13, 14] by the solution of the differential equation

$$\alpha \frac{d^2\varphi}{dx^2} = \mathbf{EP} \sin \varphi \tag{5}$$

where $\alpha = K \sin^2 \theta$ and K is the elastic constant, with the boundary conditions

$$\begin{aligned} \alpha \frac{d\varphi}{dx} &= \gamma_1 \sin 2\varphi - \gamma_2 \sin \varphi \qquad \text{at } x = 0 \text{ (bottom)} \\ \alpha \frac{d\varphi}{dx} &= -\gamma_1 \sin 2\varphi - \gamma_2 \sin \varphi \quad \text{at } x = d \text{ (top).} \end{aligned} \tag{6}$$

These equations are based on a distortion energy

$$F_d = \int_0^d \frac{1}{2} \alpha \left(\frac{d\varphi}{dx} \right)^2 dx, \tag{7}$$

an electrostatic energy

$$F_e = \frac{1}{2} \int_0^d \varepsilon\varepsilon_0 \mathbf{E}^2 \, dx \tag{8}$$

and a surface interaction energy

$$F_s = \gamma_1 (\sin^2 \varphi_b + \sin^2 \varphi_t) + \gamma_2 (\cos \varphi_b - \cos \varphi_t) \tag{9}$$

where φ_b and φ_t denote the φ-value at the bottom and top plates, respectively. The first term in F_s describes the tendency of the director to be parallel to the alignment layers ($\varphi = 0$ or π), and the second term the polar interaction with the alignment layers favoring the polarization pointing into the alignment layer ($\varphi_b = \pi$, $\varphi_t = 0$). In low-**P** material the γ_1-term is usually dominant or at least of the same order as γ_2. In [13, 14] this situation was studied and it gave rise to stationary states as represented in figure 4(*a*). In the bulk, the orientation φ is uniform but there are two surface layers with strong φ gradients. Equation (5) shows that such a uniform bulk state is only possible if $\mathbf{E} = 0$ or if $\varphi = 0$ or π. Exceptionally, that is, for very strong surface anchoring, such a state could, with increasing V, go continuously from $\varphi = 0$ to $\varphi = \pi$, but normally there is a threshold where φ jumps to π. Notice that if one tries to approximate these stationary states by the uniform theory, thus ignoring the surface layers, the polar interaction term drops out of the surface energy and only the first term remains, giving rise to a positive sign in equation (1) and thus to the well known rectangular threshold and hysteresis curve.

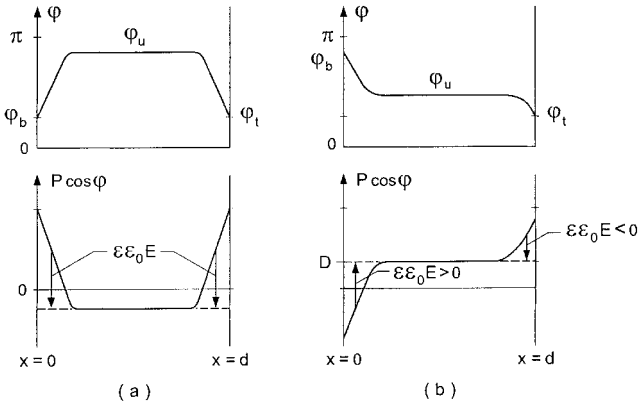

Figure 4. Orientation and polarization and field distribution in (*a*) a non-splayed state and (*b*) a splayed state.

In high-**P** materials the polar interaction energy γ_2 dominates γ_1. It is reasonably assumed to be proportional to **P** and may thus possibly be several orders of magnitude larger than γ_1. This situation was not studied in [13, 14]. We shall here analyse this situation and in the next section reduce it to the uniform theory. The solution is based on the fact that the displacement field $\mathbf{D} = \varepsilon\varepsilon_0 \mathbf{E} + \mathbf{P} \cos \varphi$ is independent of x. By multiplying equation (5) with $d\varphi/dx$ one obtains

$$\frac{\varepsilon\varepsilon_0 \alpha}{2} \frac{d}{dx} \left(\frac{d\varphi}{dx} \right)^2 = (\mathbf{D} - \mathbf{P} \cos \varphi) \frac{d}{dx} (\mathbf{D} - \mathbf{P} \cos \varphi)$$

or

$$\varepsilon\varepsilon_0 \alpha \left(\frac{d\varphi}{dx} \right)^2 = a + (\mathbf{D} - \mathbf{P} \cos \varphi)^2 \tag{10}$$

where a is a constant. We are looking for a solution which is uniform in the bulk with two surface layers which come close to $\varphi_t = 0$ and $\varphi_b = \pi$. This is a so-called splayed state. This type of solution corresponds to $a = 0$ since in the uniform bulk region $\mathbf{E} = 0$ and $d\varphi/dx = 0$. The situation is shown in figure 4(*b*). It leads to

$$\frac{d\varphi}{dx} = \pm \frac{\mathbf{P}}{(\varepsilon\varepsilon_0 \alpha)^{1/2}} (\cos \varphi - \cos \varphi_u) = \pm \xi^{-1} (\cos \varphi - \cos \varphi_u) \tag{11}$$

with + at $x = 0$ and − at $x = d$, and where we have introduced the polar coherence length $\xi = (\alpha\varepsilon\varepsilon_0/\mathbf{P}^2)^{1/2}$ first discussed in [18]. This characteristic length is a measure of the thickness of the surface layers, within which there is a considerable variation of φ. The boundary conditions described in equation (6) require

$$\cos \varphi_u = \cos \varphi_t - \frac{(\varepsilon\varepsilon_0/\alpha)^{1/2}}{\mathbf{P}} (\gamma_1 \sin 2\varphi_t + \gamma_2 \sin \varphi_t) \tag{12}$$

$$\cos \varphi_u = \cos \varphi_b - \frac{(\varepsilon\varepsilon_0/\alpha)^{1/2}}{\mathbf{P}} (\gamma_1 \sin 2\varphi_b - \gamma_2 \sin \varphi_b). \tag{13}$$

A simple computer program chooses first φ_t, calculates φ_u from equation (12) and φ_b by iteration from equation (13). The voltages over the bottom layer, the top layer and the whole liquid crystal are respectively given by

$$V_b = \left(\frac{\alpha}{\varepsilon\varepsilon_0}\right)^{1/2}(\varphi_b - \varphi_u) > 0$$
$$V_t = \left(\frac{\alpha}{\varepsilon\varepsilon_0}\right)^{1/2}(\varphi_t - \varphi_u) < 0 \qquad (14)$$
$$V = \left(\frac{\alpha}{\varepsilon\varepsilon_0}\right)^{1/2}(\varphi_b + \varphi_t - 2\varphi_u).$$

The deformation energy and the electrostatic energy are given by

$$F_d = F_e = \frac{1}{2}\mathbf{P}(\alpha/\varepsilon\varepsilon_0)^{1/2}[2\sin\varphi_u - \sin\varphi_t - \sin\varphi_b + (\varphi_t + \varphi_b - 2\varphi_u)\cos\varphi_u]. \qquad (15)$$

The surface interaction energy is given by equation (9). The voltage V in equation (14) is the voltage over the liquid crystal alone. The voltage V_a over the alignment layers should be added to it to obtain the voltage V_d over the whole display. Since $\mathbf{D}$ is a constant over the whole display and equals $\mathbf{P}\cos\varphi_u$, one obtains

$$V_d = \mathbf{P}\cos\varphi_u \frac{2d_a}{\varepsilon_a\varepsilon_0} + \mathbf{P}(\varphi_b + \varphi_t - 2\varphi_u)\frac{\xi}{\varepsilon\varepsilon_0} \qquad (16)$$

where d_a is the thickness of one alignment layer, and ξ is the polar coherence length defined in equation (11). Based on $\alpha = 4\times10^{-12}$ N and $\varepsilon = 5$, the value of ξ is, for a moderately large $\mathbf{P}$ value of 100 nC cm^{-2}, equal to 13 nm. Values of d_a are normally between 50 and 100 nm. Therefore the voltage over the alignment layers is the dominant term in equation (16), leading to the important conclusion that the width of the V-shape ($\cos\varphi_u = 1$) is proportional to $\mathbf{P}$ and to d_a.

4. Reduction to the uniform theory

Although the exact analytic solution developed in the previous section is straightforward, we think it is still much more complicated than the uniform φ theory. We will now reduce the exact theory to the uniform φ theory by rewriting the exact Gibbs energy in the form of the Gibbs energy of the uniform theory plus a correction term G_c, which is then interpreted as a fictitious interaction energy describing the effect of the surface layers.

The splayed state follows from the minimization at prescribed $\mathbf{E}(x)$ and varying $\varphi(x)$ of the Gibbs energy [14]

$$G = F_d - F_e + F_s - \int \mathbf{EP}\cos\varphi\,dx. \qquad (17)$$

We try now to relate this to the Gibbs energy in the uniform state

$$G = -\int \mathbf{E}_u\mathbf{P}\cos\varphi_u\,dx + 2\gamma_1\sin^2\varphi_u + G_c$$

where $\mathbf{E}_u = V/d$ and where G_c is a correction term. Since

$$\mathbf{D} = \varepsilon\varepsilon_0\mathbf{E} + \mathbf{P}\cos\varphi = \mathbf{P}\cos\varphi_u$$

one obtains

$$-\int \mathbf{EP}\cos\varphi\,dx = -\mathbf{P}\cos\varphi_u V + \int \varepsilon\varepsilon_0\mathbf{E}^2\,dx = -\mathbf{P}\cos\varphi_u\mathbf{E}_u d + 2F_d.$$

We have already shown that $F_d = F_e$. Therefore

$$G_c = \gamma_1(\sin^2\varphi_b + \sin^2\varphi_t - 2\sin^2\varphi_u) + \gamma_2(\cos\varphi_b - \cos\varphi_t) + \mathbf{P}\left(\frac{\alpha}{\varepsilon\varepsilon_0}\right)^{1/2}[2\sin\varphi_u - \sin\varphi_t - \sin\varphi_b + (\varphi_t + \varphi_b - 2\varphi_u)\cos\varphi_u]. \qquad (18)$$

In order to simplify the calculations, we shall from now on set $\gamma_1 = 0$ and introduce

$$g = \frac{\gamma_2}{\mathbf{P}}\left(\frac{\varepsilon\varepsilon_0}{\alpha}\right)^{1/2} = 3.3\frac{\gamma_2}{\mathbf{P}}. \qquad (19)$$

The numerical value is based on $\varepsilon = 5$ and $\alpha = 4.10^{-12}$ N. We examine G_c for φ_u varying from $\pi/2$ to 0. The first limiting case corresponds to

$$V = 0, \quad \mathrm{tg}\left(\frac{\pi}{2} - \varphi_t\right) = \mathrm{tg}\left(\varphi_b - \frac{\pi}{2}\right) = g, \quad \varphi_u = \pi/2 \qquad (20)$$

and the second limiting case to

$$V = V_m = \left(\frac{\alpha}{\varepsilon\varepsilon_0}\right)^{1/2}\varphi_b, \quad \mathrm{tg}\frac{\varphi_b}{2} = g, \quad \varphi_u = \varphi_t = 0. \qquad (21)$$

Both cases are shown in figure 5.

In figure 6 we have shown G_c/γ_2 as a function of φ_u for $g = 1$. We find a qualitatively excellent approximation in

$$G_c = const - \gamma\sin^2\varphi_u. \qquad (22)$$

For other values of g one obtains similar results. We can identify γ from the amplitude of the $\sin^2\varphi_u$ behaviour.

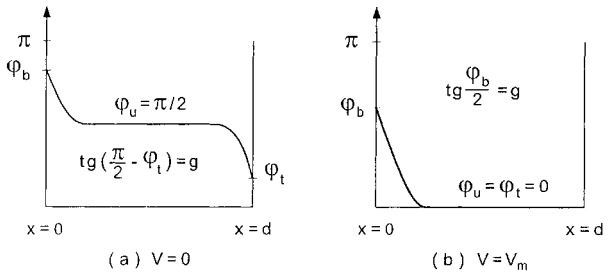

Figure 5. Splayed state at (*a*) $V = 0$, and (*b*) $V = V_m$.

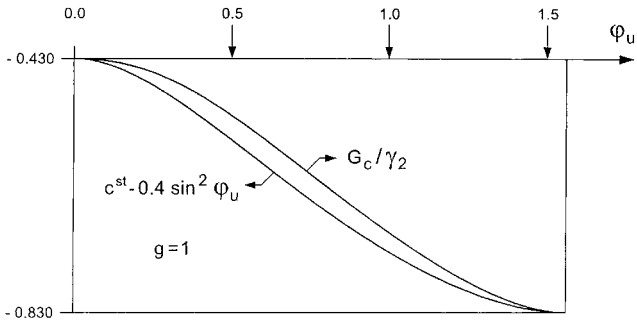

Figure 6. Reduction of splayed state to uniform state through the approximation of G_c by *const.* $-\gamma \sin^2 \varphi_u$.

In the table presented below γ/γ_2 is calculated for various values of g. On the other hand, one could estimate γ from the behaviour of the function $\sin^2 \varphi_u$ near $\varphi_u = 0$, i.e. from identifying V_m where the state $\varphi_u = 0$ is reached with the V_m from the uniform theory:

$$V_m = \frac{2\gamma}{P} = \left(\frac{\alpha}{\varepsilon\varepsilon_0}\right)^{1/2} \varphi_b \quad \text{or} \quad \frac{\gamma}{\gamma_2} = \frac{\varphi_b}{2g}. \tag{23}$$

Both estimates are shown in the table. The fact that they do not coincide illustrates that the reduction of the splayed state to the uniform theory is only qualitatively correct.

5. Stability

The uniform theory leads to a very simple stability criterion: the second term in the right hand side of equation (1) has to be negative. In the exact theory of the splayed state, variational theory is vague and incomplete about stability. We derive here our own weak criterion by which we investigate whether a neighbouring stationary solution will return to the original stationary solution under its original voltage. It is certainly a necessary condition for stability but certainly not a sufficient condition for stronger stability criteria. It is based on the equation of motion of a non-stationary state

$$\eta \frac{\partial \varphi}{\partial t} = \alpha \frac{\partial^2 \varphi}{\partial x^2} - \mathbf{EP} \sin \varphi.$$

Let us consider a stationary solution φ_1, requiring an electric field distribution $\mathbf{E}_1$ and a corresponding voltage V_1. Let us also consider a neighbouring stationary solution φ_2, requiring an electric field distribution $\mathbf{E}_2$ and a corresponding voltage V_2. We shall prove that if $V_2 > V_1$ implies $\varphi_2 < \varphi_1$ for all x, then the solution φ_1 is stable. Let us indeed investigate how the solution φ_2 behaves under the influence of V_1. Thus at time $t = 0$, the initial configuration is $\varphi_2(x)$ for voltage V_2, field $\mathbf{E}_2$ and displacement field $\mathbf{D}_0 = \varepsilon\varepsilon_0 \mathbf{E}_2 + \mathbf{P} \cos \varphi_2$. Then at time $t = 0^+$, just after $t = 0$, the voltage is immediately changed to V_1 and the field changes to $\mathbf{E}$, but the director remains at φ_2 so that the displacement field is $\mathbf{D}_{0+} = \varepsilon\varepsilon_0 \mathbf{E} + \mathbf{P} \cos \varphi_2$. Since $\mathbf{D} = \varepsilon\varepsilon_0 \mathbf{E} + \mathbf{P} \cos \varphi$ is independent of x, this electric field $\mathbf{E}$ can only differ from $\mathbf{E}_2$ by a constant independent of x. Therefore

$$\begin{aligned} \eta \frac{\partial \varphi_2}{\partial t} &= \alpha \frac{\partial^2 \varphi_2}{\partial x^2} - \mathbf{EP} \sin \varphi_2 \\ &= \left[\alpha \frac{\partial^2 \varphi_2}{\partial x^2} - \mathbf{E}_2 \mathbf{P} \sin \varphi_2 \right] + (\mathbf{E}_2 - \mathbf{E}) \mathbf{P} \sin \varphi_2 \\ &= (\mathbf{E}_2 - \mathbf{E}) \mathbf{P} \sin \varphi_2. \end{aligned}$$

The term between square brackets is indeed zero since φ_2 was supposed to be a stationary solution under $\mathbf{E}_2$. If thus $\mathbf{E}_2 > \mathbf{E}$, i.e. if $V_2 > V_1$, it follows that φ_2 will increase in time and thus approach φ_1. The stability condition is thus

$$\frac{\mathrm{d}V}{\mathrm{d}\varphi} < 0 \tag{24}$$

or in words: if V increases, φ at any x decreases. That this is so, follows from the fact that if φ_t (or φ_b) decreases, φ_u decreases even more and thus V increases. The proof of this last statement is not straightforward, but follows from smart manipulations of the equations. It is omitted here.

Table. γ/γ_2 as estimated from the amplitude of $\sin^2 \varphi_u$ or from the value of V_m.

g	γ_{ampl}/γ_2	γ_{Vm}/γ_2
$1/\sqrt{3}$	0.41	0.91
1	0.40	0.79
$\sqrt{3}$	0.36	0.61
2	0.34	0.55
3	0.28	0.42
4	0.23	0.33
5	0.19	0.27

6. Stationary states for $V > V_m$

At V_m the splayed state reaches $\varphi_u = 0$ in the bulk. For higher voltages, this uniform part is no longer field

free. It is now governed by equation (10) but with a different from zero, i.e.

$$\varepsilon\varepsilon_0\alpha\left(\frac{d\varphi}{dx}\right)^2 = (\mathbf{D} - \mathbf{P}\cos\varphi)^2 - (\mathbf{D} - \mathbf{P})^2$$

$$= 4\mathbf{P}^2 \sin^2\frac{\varphi}{2}\left(\frac{\mathbf{D}}{\mathbf{P}} - \cos^2\frac{\varphi}{2}\right) \qquad (25)$$

as shown in figure 7. This equation can be solved analytically, e.g. by a mathematical manipulation package. We will give here the main intermediate steps to make it possible for the reader to follow the calculations.

The boundary condition of the first of the equations (6) with $\gamma_1 = 0$ leads, after some calculations, to

$$\cos^2\frac{\varphi_b}{2}(1 + g^2) = \frac{\mathbf{D}}{\mathbf{P}} \qquad (26)$$

and the potential V is given by

$$V = \int_0^d \mathbf{E}\,dx = \frac{(\mathbf{D} - \mathbf{P})d}{\varepsilon\varepsilon_0} + \int_{\varphi_b}^0 \frac{\mathbf{P}}{\varepsilon\varepsilon_0}(1 - \cos\varphi)\frac{d\varphi}{d\varphi/dx}$$

or, again after some calculations using the previous equation and equation (25), by

$$V = \frac{\mathbf{D} - \mathbf{P}}{\varepsilon\varepsilon_0}d - 2\left(\frac{\alpha}{\varepsilon\varepsilon_0}\right)^{1/2} \text{Arc sin}\,\frac{\cos\varphi_b/2}{(\mathbf{D}/\mathbf{P})^{1/2}}$$

$$+ 2\left(\frac{\alpha}{\varepsilon\varepsilon_0}\right)^{1/2} \text{Arc sin}\,\frac{1}{(\mathbf{D}/\mathbf{P})^{1/2}}. \qquad (27)$$

If we introduce the value of φ_b at $V = V_m$, i.e

$$\text{tg}\frac{\varphi_{bm}}{2} = g \quad \text{or} \quad \cos\frac{\varphi_{bm}}{2} = \frac{1}{(1 + g^2)^{1/2}} \qquad (28)$$

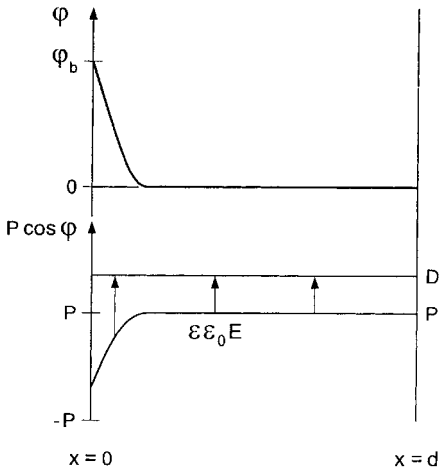

Figure 7. φ distribution for $V > V_m$.

we can rewrite the boundary condition (26) and the voltage (27) as

$$\frac{\mathbf{D}}{\mathbf{P}} = \frac{\cos^2\varphi_b/2}{\cos^2\varphi_{bm}/2} \qquad (29)$$

$$V = \left(\frac{\alpha}{\varepsilon\varepsilon_0}\right)^{1/2}\left[\frac{\mathbf{P}d}{(\varepsilon\varepsilon_0\alpha)^{1/2}}\left(\frac{\mathbf{D}}{\mathbf{P}} - 1\right)\right.$$

$$\left. + 2\,\text{Arc sin}\,\frac{\cos\varphi_{bm}/2}{\cos\varphi_b/2} - (\pi - \varphi_{bm})\right]. \qquad (30)$$

The behaviour of V as a function of φ_b is shown in figure 8. Careful checking around V_m shows that there is a very small threshold and hysteresis jump. Theoretically the decrease of V with decreasing φ_b derives from the disappearance of the top surface layer. The small threshold jump and hysteresis jumps cause jumps in the bottom surface layer, but no breaking down of this layer. The practical significance is minor.

The value of the dimensionless parameter $b = \mathbf{P}d/(\varepsilon\varepsilon_0\alpha)^{1/2} = d/\xi$ can be estimated to vary between 45 and 450 for $d = 2\,\mu\text{m}$ and $\mathbf{P}$ varying from 30 to 300 nC cm^{-2}. The latter value corresponds to a boundary layer of about 5 nm, i.e. about the size of a single molecule.

7. Conclusion

Grey levels in ferroelectric or antiferroelectric liquid crystal displays can be obtained either through 'in-pixel domain switching', or through 'continuous director rotation'. The latter can only be used with active matrix addressing. We have shown that for this latter case there is a simple analytic theory based on the uniform φ theory in a bookshelf structure, which means that the azimuthal angle φ on the cone is independent of the depth into the liquid crystal. The theory requires a physical explanation for the fact that the equilibrium state at zero applied voltage gives rise to a polarization parallel to the glass plates, which also means that the director makes a maximum angle, i.e. the cone angle,

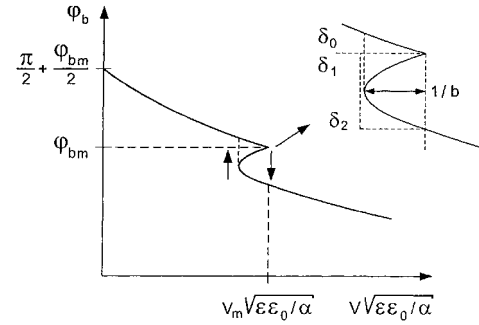

Figure 8. Schematic representation of φ_b versus V, showing the unstable region where $dV/d\varphi_b > 0$. The values of $\delta = |\varphi_b - \varphi_{bm}|$ are, respectively $\delta_0 = 1/8gb^2$, $\delta_1 = (1/gb^2)[1/(1 + g^2)]$, $\delta_2 = (4/gb^2)[1/(1 + g^2)]$.

with these glass plates. In previous articles this theory was applied to describe the 'V-shaped' characteristics in antiferroelectric liquid crystal displays [8] and the CDR-mode in low-**P** ferroelectric liquid crystals [10]. In this paper we have concentrated on the analytic description of the V-shaped characteristic in high-**P** ferroelectric liquid crystals explained by the Lagerwall and Clark group as a splayed state [4, 11, 12]. This group mainly concentrates on qualitative physical arguments to explain the untwisting in the bulk of a splayed ferroelectric liquid crystal. We concentrated on the exact analytic analysis of the splayed state. Within the width of the V-shape ($|V| < V_m$), the analysis is simple and straightforward. We also showed that the voltage over the alignment layers is dominant over the voltage over the liquid crystal and confirmed the important result that this dominant voltage is proportional to the value of **P** and to the thickness of the alignment layer. In the saturation region ($|V| > V_m$) the exact analysis becomes rather complicated and also the stability criterion is very difficult to verify. We therefore reduced the exact theory to the extremely simple uniform φ theory by introducing a fictitious surface interaction energy that gives rise to the Gibbs energy of the exact theory. It turns out that this fictitious interaction energy was excellently approximated by the second term in equation (1), with the right sign of γ. Therefore the uniform theory could be applied: the splayed state gives rise to a uniform equilibrium state at $\varphi = \pi/2$ which is then continously pulled to $\varphi = 0$ by the electric field at a voltage V_m. Beyond V_m it stays in the state $\varphi = 0$. Of course the electric field distribution is not the same (it is uniform). Also the voltage over the liquid crystal is only qualitatively correct but since this is not the dominant voltage, this is not very important. Also the stability did not exactly correspond: the uniform theory gives rise to a stable state, the exact theory has a very small range of unstable states near V_m. These differences are of minor importance, and the main advantage of the uniform theory remains in its tremendous simplicity.

This work was carried out in the framework of a European network for optical research on chiral systems (Orchis), contract no ERBXCT97-0119. The authors acknowledge A. De Vos for careful checking of the small unstable range for $V \geqslant V_m$.

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